# Research record #4

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by

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# Introduction

Both Fubuki and Hayate is a Sun SOL2 computer furnated with MATLAB Version 5.2.1.1420 located at the Department of Control System T.I.T.

# DC motor

## Field-controlled DC motor

Consider a field-control DC motor [Kha96]

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \tag{1}$$

$$v_a = c_1 i_f \omega + L_a \frac{di_a}{dt} + R_a i_a \tag{2}$$

$$J \frac{d\omega}{dt} = c_2 i_f i_a - c_3 \omega, \tag{3}$$

where

**Equation 1** is the equation of the field circuit. Here $v_f$, $i_f$, $R_f$, and $L_f$ are respectively voltage, current, resistance, and inductance of the field circuit. The input to the system is $v_f$.

**Equation 2** is the equation of the armature circuit. Here $v_a$, $i_a$, $R_a$, and $L_a$ are respectively voltage, current, resistance, and inductance of the armature circuit. Moreover

$$\begin{aligned} c_1 i_f \omega &= \text{the back e.m.f. induced in the armature circuit} \\ v_a &= \text{constant} \end{aligned}$$

**Equation 3** is a torque equation of the shaft where

$$\begin{aligned} J &= \text{the rotor inertia} \\ c_3 &= \text{damping coefficient} \\ c_2 i_f i_a &= \text{the torque produced by the interaction of the armature current with the field circuit flux} \end{aligned}$$

Equation 1 to 3 can be rearranged into

$$\frac{di_f}{dt} = -\frac{R_f}{L_f} i_f + \frac{v_f}{L_f}. \tag{4}$$

$$\frac{di_a}{dt} = -\frac{c_1}{L_a} i_f \omega - \frac{R_a}{L_a} i_a + \frac{v_a}{L_a} \tag{5}$$

$$\frac{d\omega}{dt} = \frac{c_2}{J} i_f i_a - \frac{c_3}{J} \omega \tag{6}$$

## Constant input

If one lets [1]

$$v_a = V_a = \text{constant}, \quad v_f = U = \text{constant}$$

then the unique equilibrium point is at

$$I_a = \frac{c_3 R_f^2 V_a}{c_3 R_a R_f^2 + c_1 c_2 U^2}, \quad \Omega = \frac{c_2 R_f U V_a}{c_3 R_a R_f^2 + c_1 c_2 U^2}, \quad I_f = \frac{U}{R_f}.$$

Typically the time constants

$$T_a = \frac{L_a}{R_a} \ll T_f = \frac{L_f}{R_f} \quad \text{and} T_a \ll T_{\text{mechanical}}.$$

Then letting

$$x_1 = \frac{i_f}{I_f}, \quad x_2 = \frac{\omega}{\Omega}, \quad z = \frac{i_a}{I_a}, \quad u = \frac{v_f}{U}, \quad \varepsilon = \frac{T_a}{T_f},$$

$$\text{and use} \qquad \tau = \frac{t}{T_f} \qquad \text{as time variable}$$

then a singularly perturbed model is obtained as

$$\frac{x_1}{d\tau} = -x_1 + u \tag{7}$$

$$\frac{x_2}{d\tau} = a\,(x_1 z - x_2) \tag{8}$$

$$\varepsilon \frac{z}{d\tau} = -z - b x_1 x_2 + c \tag{9}$$

where

$$a = \frac{L_f c_3}{R_f J}, \quad b = \frac{c_1 c_2 U^2}{c_3 R_a R_f^2}, \quad c = \frac{V_a}{I_a R_a}.$$

[1] see [Kha96], Exercise 9.20

# Armature-controlled DC motor

An armature-controlled DC motor can be described by [Kha96]

$$J\frac{d\omega}{dt} = ki \tag{10}$$

$$L\frac{di}{dt} = -k\omega - Ri + u \tag{11}$$

where $i$, $u$, $R$, $L$, and $J$ are respectively armature current, voltage, resistance, inductance, and moment of inertia. Moreover,

$$\left.\begin{array}{rcl} ki & = & \text{torque} \\ k\omega & = & e.m.f.(\text{electromotive force}) \end{array}\right\} \text{developed with constant excitation flux } \phi.$$

If one neglects $L$, which is usually small, Equation 10 and 11 lead to the commonly used $1^{st}$-order model of the DC motor

$$J\dot{\omega} = \frac{k^2}{R}\omega + \frac{k}{R}u. \tag{12}$$

However if one takes $L$ into account and introduces the dimensionless variables

$$\omega_r = \frac{\omega}{\Omega}; \quad i_r = \frac{iR}{k\Omega}; \quad u_r = \frac{u}{k\Omega}$$

one obtains the state equations

$$\frac{d\omega_r}{dt_r} = i_r \tag{13}$$

$$\frac{T_e}{T_m}\frac{di_r}{dt_r} = -\omega_r - i_r + u_r, \tag{14}$$

where
$$T_m = \frac{JR}{k^2} \quad \text{is the mechanical time constant}$$
$$T_e = \frac{L}{R} \quad \text{is the electrical time constant}$$
$$t_r = \frac{t}{T_m} \quad \text{is the dimensionless time variable (justified because } T_m \gg T_e)$$
$$\varepsilon = \frac{T_e}{T_m} = \frac{Lk^2}{JR^2} \quad \text{is the perturbing parameter}$$

# Singular Perturbation with varying $\varepsilon$

Consider a system in the form

$$E\dot{x} = Ax + Bu, \tag{15}$$

where $E$ contains some parameters $\varepsilon$ which are usually small and whose values may change.

# System with signum function

$27^{th}$ **May 1998**
Consider the system

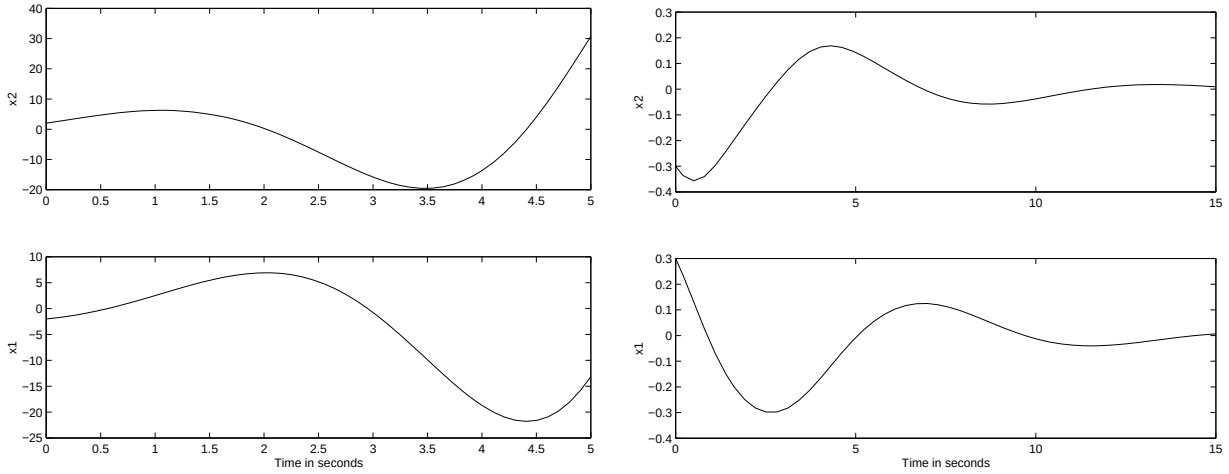
$$\dot{x_1} = x_2 \tag{16}$$
$$\dot{x_2} = -x_2 - \psi\,(x_1 - x_2) + u, \tag{17}$$

where

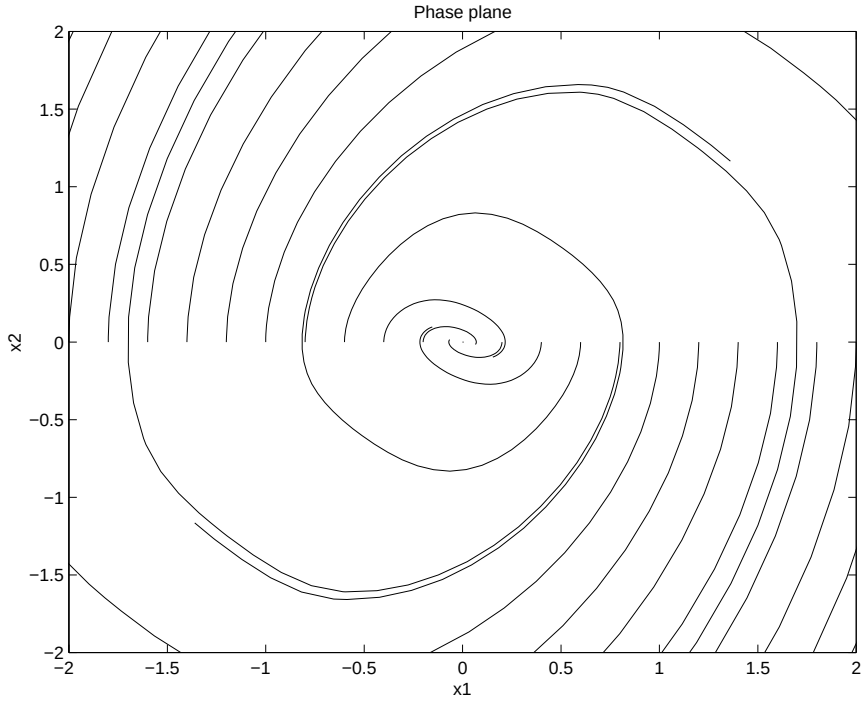
$$\psi(x) = \begin{cases} x^3 + 0.5\,x, & \text{if } |x| < 1 \\ 2\,x - 0.5\,\mathrm{sgn}(x), & \text{if } |x| \geq 1 \end{cases} \tag{18}$$

The response of this system when $u = 0$ is shown in Figure 1.

**For Figure 1** $t_{simulation} = 5$ sec, $t_{CPU,average} = 0.0100$ sec. Initial conditions are $(x_1, x_2) = (-2, 2)$.
File size =9.48 KB




(a) (b)




(c)

Figure 1: *(a) state variables, (b) state variables* ($t_{simulation} = 15$ sec, $t_{CPU} = 0.0300$ sec, *Initial conditions are* $(x_1, x_2) = (0.3, -0.3)$ *(File size =8.53 KB)), and (c) the state plane* ($t_{simulation} = 5$ sec, $t_{CPU,average} = 0.0200$ sec, *Initial conditions are* $(x_1, x_2) \in \{-3, -2.8, -2.6, \ldots, 3\} \times \{0\}$ *(File size =12.54 KB)), simulated on Hayate*

Figure 2 shows the result when $u = -3\,\mathrm{sgn}(x_1 + x_2)$ is applied. When simulating for Figure 2 (a) and (c) the simulation was stopped manually to optimize the use of computer resource.

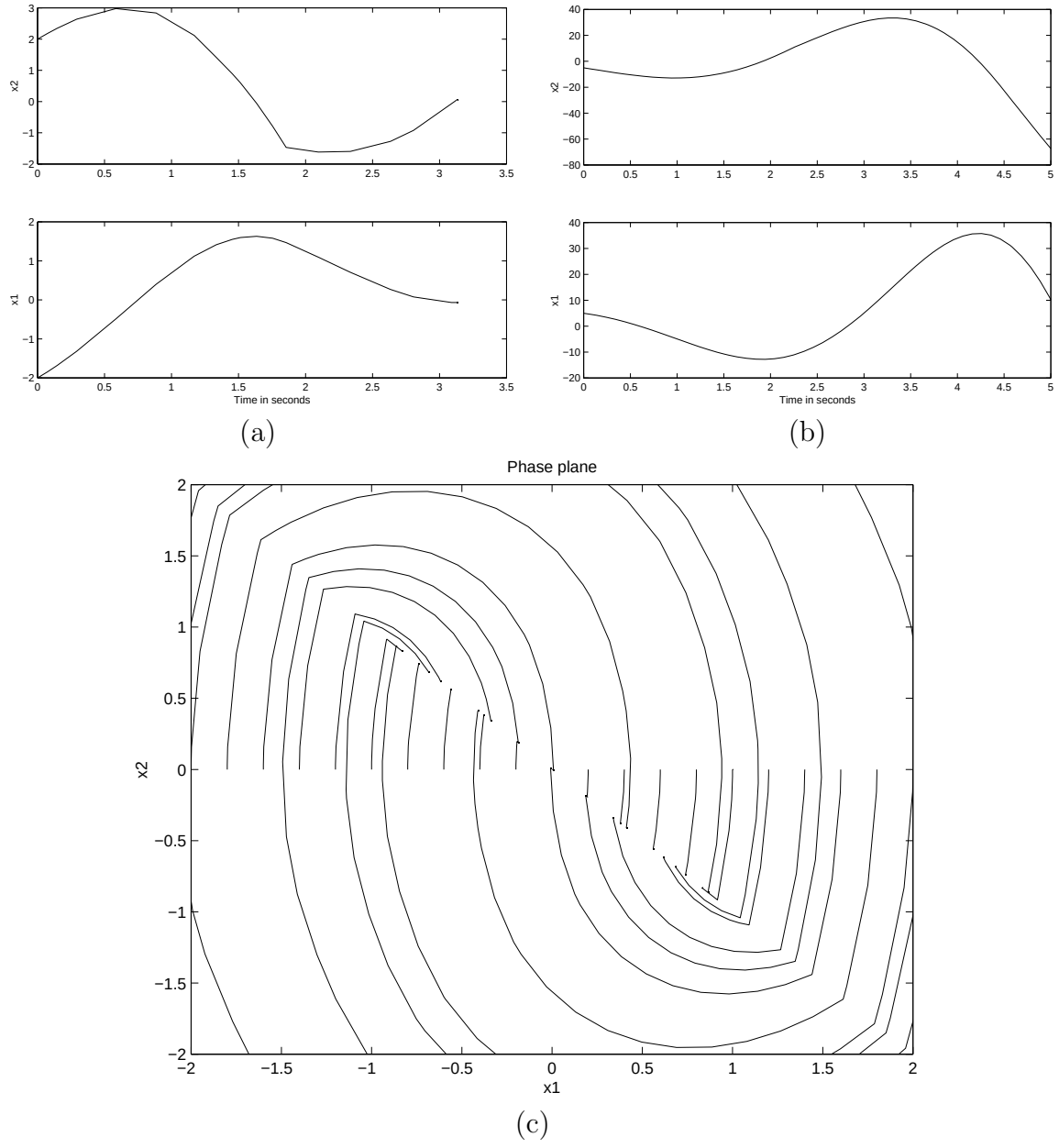



Figure 2: *(a) The state variables ($t_{CPU} = 10.3400$ sec, Initial conditions are $(x_1, x_2) = (-2, 2)$ (File size =300.72 KB)), (b) The state variables ($t_{simulation} = 5$ sec, $t_{CPU} = 0.0300$ sec, Initial conditions are $(x_1, x_2) = (5, -5)$ (File size =9.50 KB)), and (c) the state plane ($t_{CPU} = 53.8300$ sec, Initial conditions are $(x_1, x_2) \in \{-3, -2.8, -2.6, \ldots, 3\} \times \{0\}$ (File size =817.75 KB)), simulated on Hayate*

Try again with a larger $k$. Let $k = -15$ (ie $u = -15\,\mathrm{sgn}(x_1 + x_2)$) and plot Figure 3 as the result.

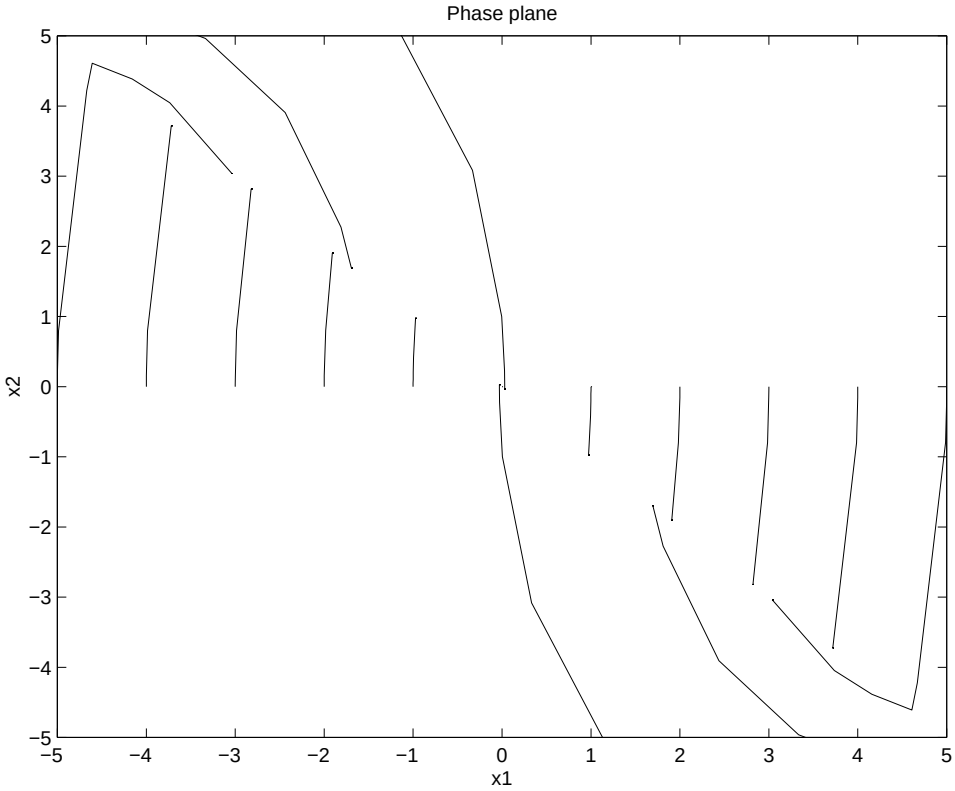


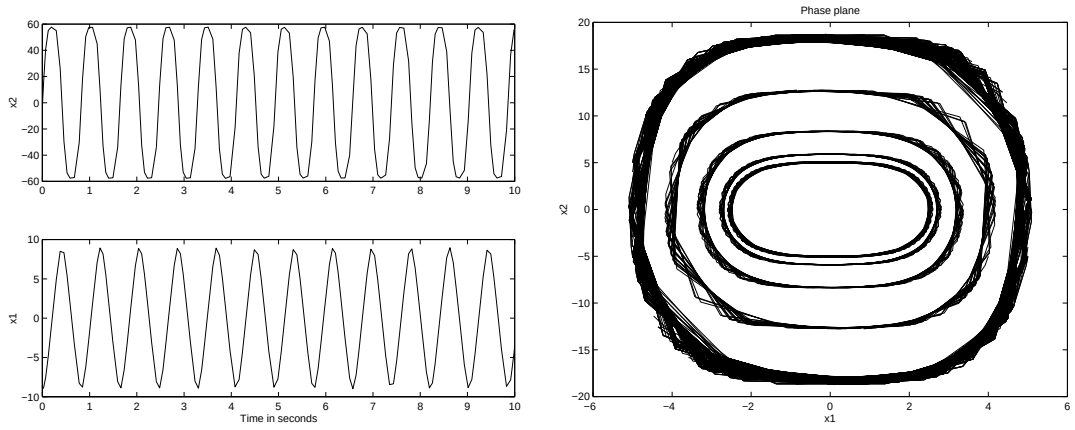

Figure 3: *State plane ($t_{CPU}$ = 23.1600 sec, Initial conditions are $(x_1, x_2) \in \{-7 \leq \mathcal{I} \leq 7\} \times \{0\}$ (File size =334.47 KB)), simulated on Hayate*

# Second order state-space with cubic term

$26^{th}$ **May 1998**
Consider the system

$$\dot{x_1} = x_2 \tag{19}$$
$$\dot{x_2} = -\left(0.5\,x_1 + x_1^3\right) + u, \tag{20}$$

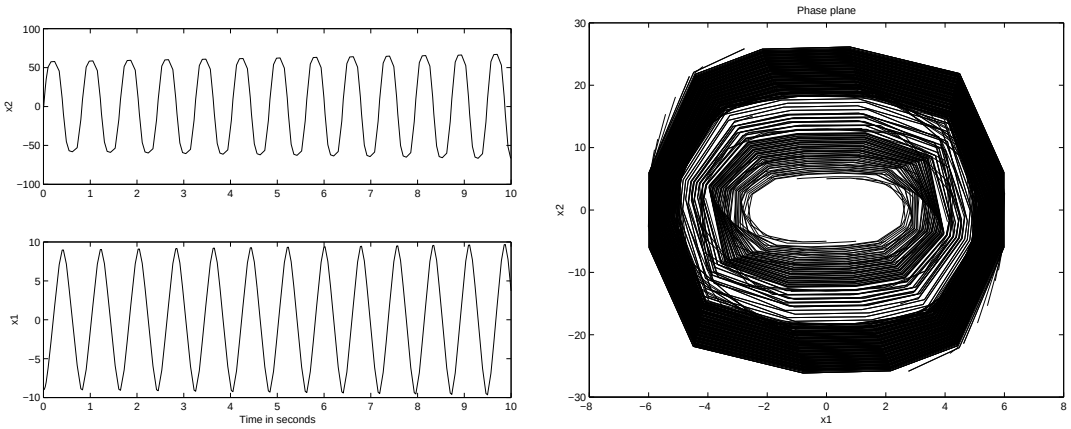
which when $u = 0$ produces Figure 4.




(a) (b)

Figure 4: *(a) The state variables ($t_{simulation} = 10$ sec, $t_{CPU} = 0.0200$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{simulation} = 10$ sec, $t_{CPU,average} = 0.0111$ sec, Initial conditions are $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = 0$, simulated on Hayate*

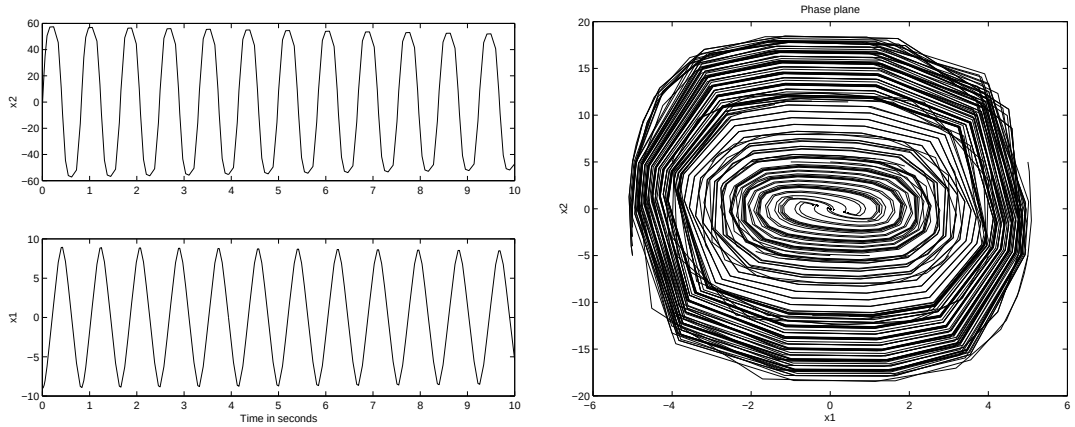
Let $u = \mathrm{sgn}\,(x_1 + x_2)$ and obtain Figure 5.




(a) (b)

Figure 5: *(a) The state variables* ($t_{simulation} = 10$ sec, $t_{CPU} = 0.0300$ sec, *Initial conditions are* $(x_1, x_2) = (-9, -3)$*) (b) the state plane* ($t_{simulation} = 10$ sec, $t_{CPU,average} = 0.0207$ sec, *Initial conditions are* $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$*)* $u = \mathrm{sgn}\,(x_1 + x_2)$*, simulated on Hayate*

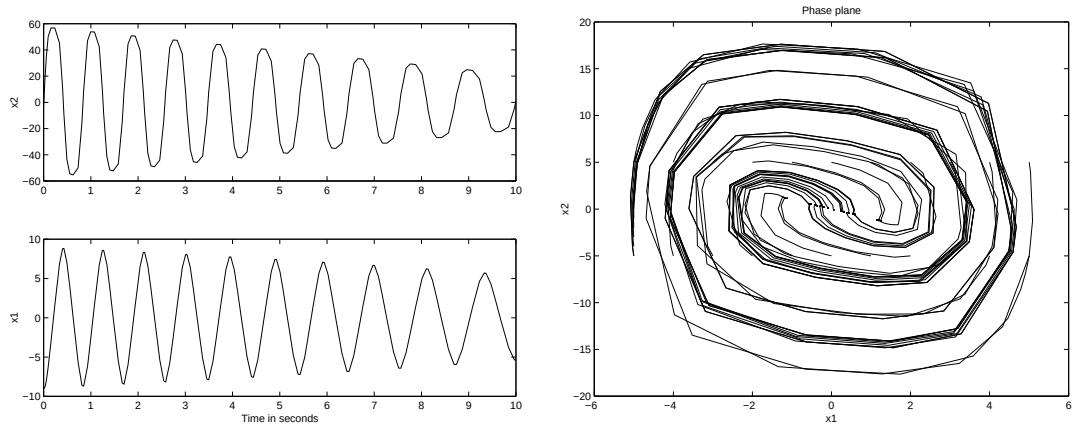
Next let $u = -\text{sgn}\,(x_1 + x_2)$ and again obtain Figure 6. Used manual interrupts to prevent the simulation from using up unnecessary computer resources.




(a) (b)

Figure 6: *(a) The state variables ($t_{CPU} = 0.0300$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{CPU} = 22.0700$ sec, Planned initial conditions were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = -\text{sgn}\,(x_1 + x_2)$, simulated on Hayate*

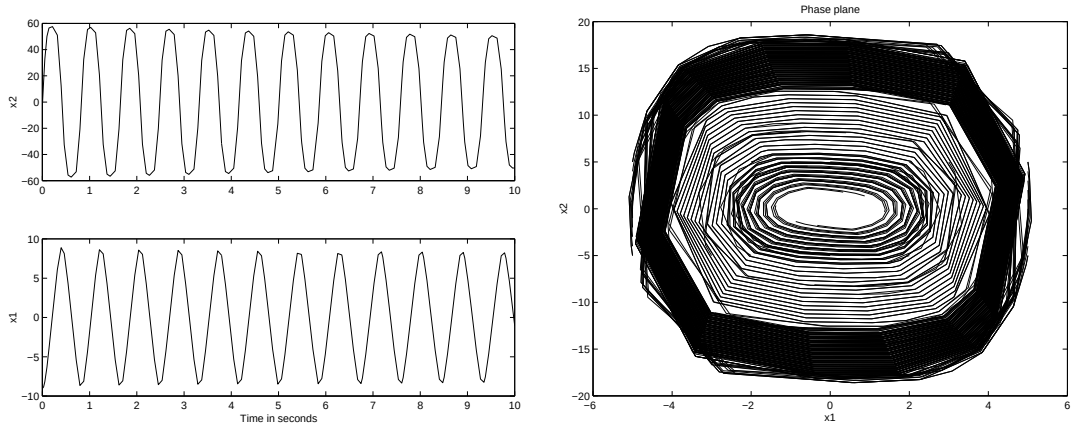
Change the feedback gain to $k = -5$ and let $u = -5\text{sgn}\,(x_1 + x_2)$ and again obtain Figure 7. Used manual interrupts to prevent the simulation from using up unnecessary computer resources (but only in Figure 7 (b) case, not Figure 7 (a)'s).




(a) (b)

Figure 7: *(a) The state variables ($t_{simulation} = 10$ sec$, t_{CPU} = 0.0300$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{CPU} = 35.9700$ sec, Planned initial conditions were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = -5\text{sgn}\,(x_1 + x_2)$, simulated on Hayate*

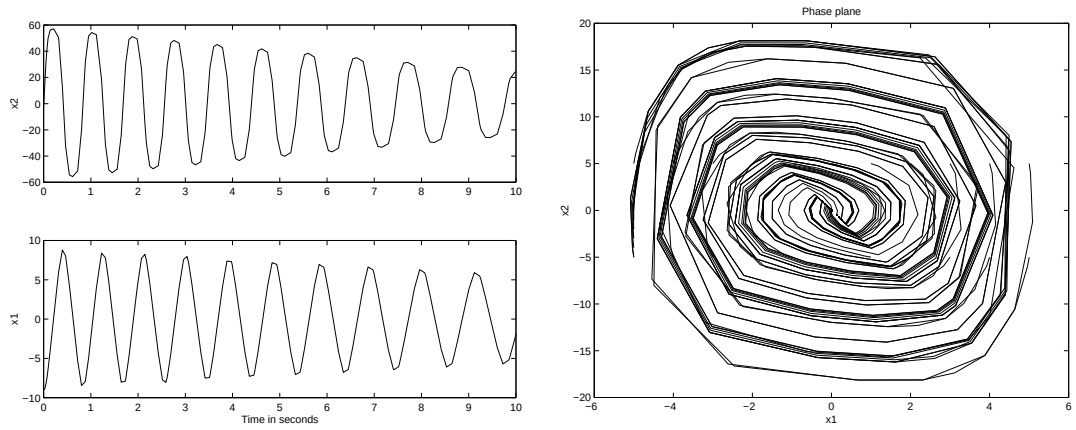
This time change the slope of the hyperplane by letting $u = -\mathrm{sgn}\,(4\,x_1 + x_2)$ and again obtain Figure 8.




(a) (b)

Figure 8: *(a) The state variables ($t_{simulation} = 10$ sec, $t_{CPU} = 0.0200$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{simulation} = 10$ sec, $t_{CPU} = 0.0200$ sec, Initial conditions were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = -\mathrm{sgn}\,(4\,x_1 + x_2)$, simulated on Hayate*

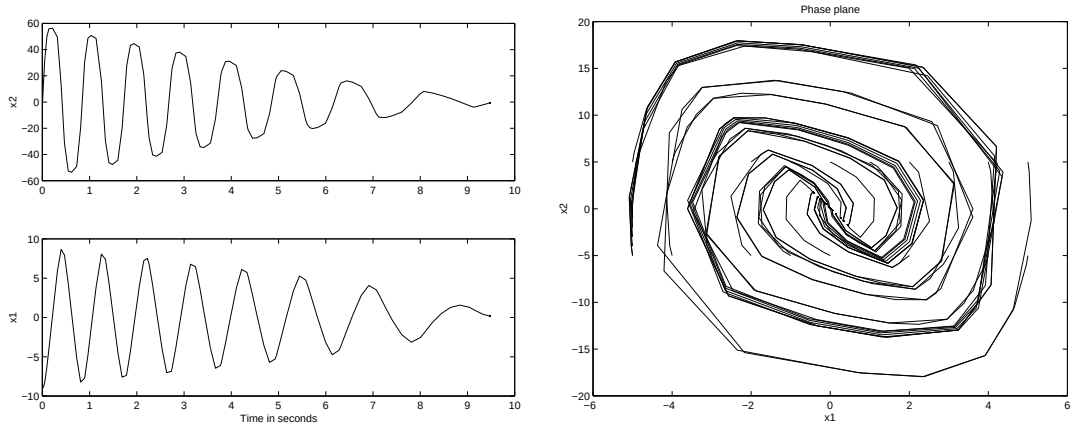
Now let $u = -5\mathrm{sgn}\,(4\,x_1 + x_2)$ and again obtain Figure 9. Manual interrupts were applied (but only when simulating for Figure 9(b), not for Figure 9 (a) case where the simulation could complete properly) when simulating in order to prevent the simulation from wasting unnecessary computer resources. The result shows both jumpings and slidings which occured on the $4\,x_1 + x_2 = 0$ hyperplane.




(a) (b)

Figure 9: *(a) The state variables ($t_{simulation} = 10$ sec, $t_{CPU} = 0.0300$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{CPU} = 36.4600$ sec, Planned initial conditions were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = -10\mathrm{sgn}\,(4\,x_1 + x_2)$, simulated on Hayate*

Then let $u = -10\mathrm{sgn}\,(4\,x_1 + x_2)$ and again obtain Figure 10. Manual interrupts were applied when simulating to prevent the simulation from wasting unnecessary computer resources.




(a) (b)

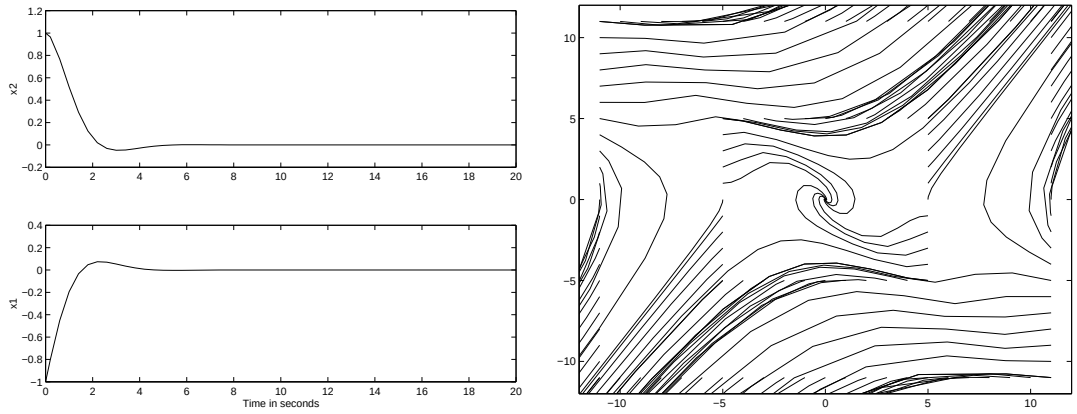
Figure 10: *(a) The state variables ($t_{CPU} = 2.0200$ sec, Initial conditions are $(x_1, x_2) = (-9, -3)$) (b) the state plane ($t_{CPU} = 25.3200$ sec, Planned initial conditions were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$) $u = -10\mathrm{sgn}\,(4\,x_1 + x_2)$, simulated on Hayate*

# Second order state-space with a trigonometric tan

$24^{th}$ **May 1998**
Consider the system

$$\dot{x_1} = x_2 \tag{21}$$
$$\dot{x_2} = x_1 + x_2 - \tan^{-1}(x_1 + x_2) + u, \tag{22}$$

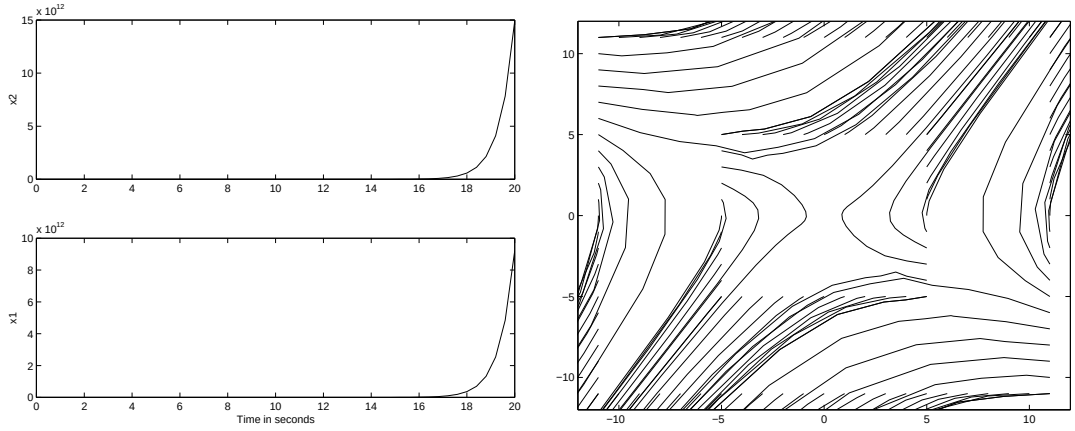
the result when $u = 0$ is shown in Figure 11.




(a) (b)

Figure 11: *(a) The state variables ($t_{simulation} = 20$ sec, $t_{CPU} = 0.0100$ sec, Initial conditions are $(x_1, x_2) = (-1, 1)$) (b) the state plane ($t_{simulation} = 20$ sec, $t_{CPU,average} = 0.0093$ sec, Initial conditions are $(x_1, x_2) \in \{-11 \leq \mathcal{I} \leq 11\} \times \{-11 \leq \mathcal{I} \leq 11\} \bigcup \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$), simulated on Hayate*

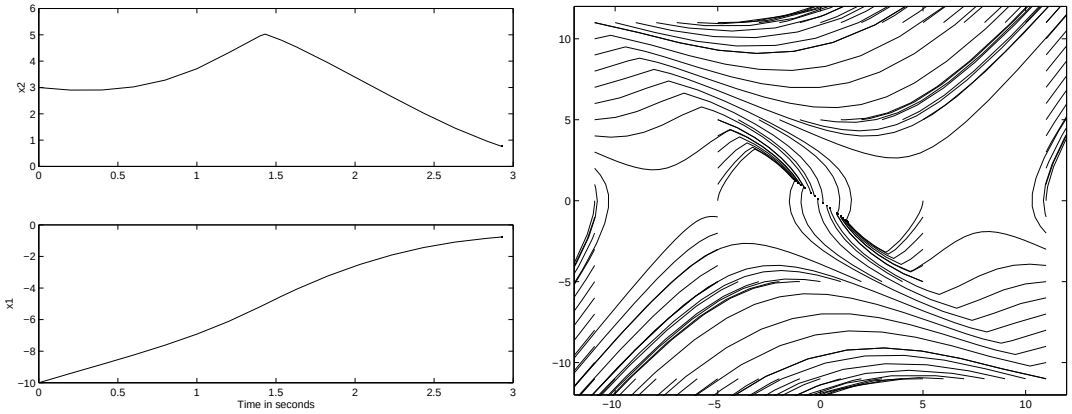
Let $u = 2\,\mathrm{sgn}(x_1 + x_2)$ and plot Figure 12.




(a) (b)

Figure 12: *(a) The state variables ($t_{simulation} = 20$ sec, $t_{CPU} = 0.0100$ sec, Initial conditions are $(x_1, x_2) = (-1, 1)$) (b) the state plane ($t_{simulation} = 20$ sec, $t_{CPU,average} = 0.0121$ sec, Initial conditions are $(x_1, x_2) \in \{-11 \leq \mathcal{I} \leq 11\} \times \{-11 \leq \mathcal{I} \leq 11\} \bigcup \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$), $u = 2\,\mathrm{sgn}(x_1 + x_2)$ simulated on Hayate*

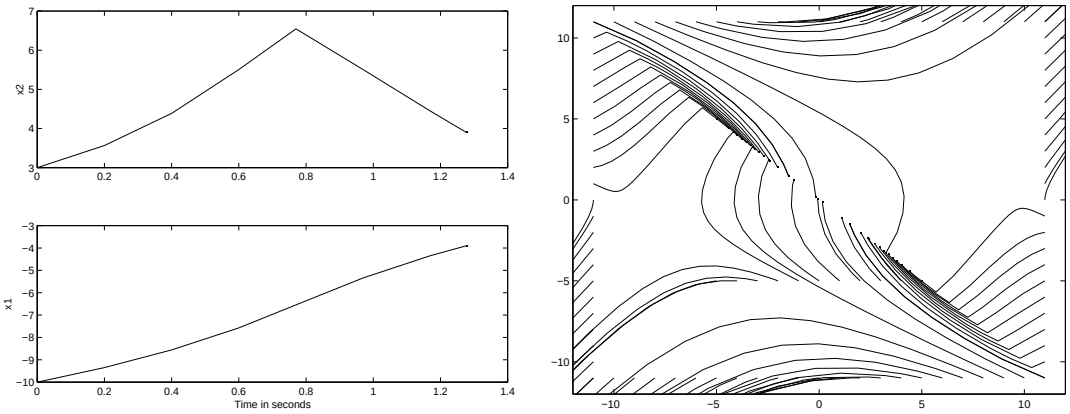
Let $u = -2\ \mathrm{sgn}(x_1 + x_2)$ and plot Figure 13. Simulation took a very long time so was stopped before the actual planned $t_{simulation} = 20$ sec was reached. Likewise not all the initial conditions planned had been simulated.




(a) (b)

Figure 13: *(a) The state variables ($t_{CPU} = 2.7100$ sec, Initial conditions are $(x_1, x_2) = (-1, 1)$) (b) the state plane ($t_{CPU} = 47.7100$ sec, Initial conditions are $(x_1, x_2) \in \{-11 \leq \mathcal{I} \leq 11\} \times \{-11 \leq \mathcal{I} \leq 11\} \bigcup \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$), $u = -2\ \mathrm{sgn}(x_1 + x_2)$ simulated on Hayate*

Next to observe the effect of the feedback gain $k$ let $u = -5\ \mathrm{sgn}(x_1 + x_2)$ and plot Figure 14, or $k = -5$. Compare this figure with Figure 13 where $k = -2$. Simulation took a very long time so was stopped before the actual planned $t_{simulation} = 20$ sec was reached. Likewise not all the initial conditions planned had been simulated.




(a) (b)

Figure 14: *(a) The state variables ($t_{CPU} = 2.1300$ sec, Initial conditions are $(x_1, x_2) = (-10, -3)$) (b) the state plane ($t_{CPU} = 56.6400$ sec, Initial conditions are $(x_1, x_2) \in \{-11 \leq \mathcal{I} \leq 11\} \times \{-11 \leq \mathcal{I} \leq 11\} \bigcup \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\}$), $u = -5\ \mathrm{sgn}(x_1 + x_2)$ simulated on Hayate*

# Second order system with limit-cycles

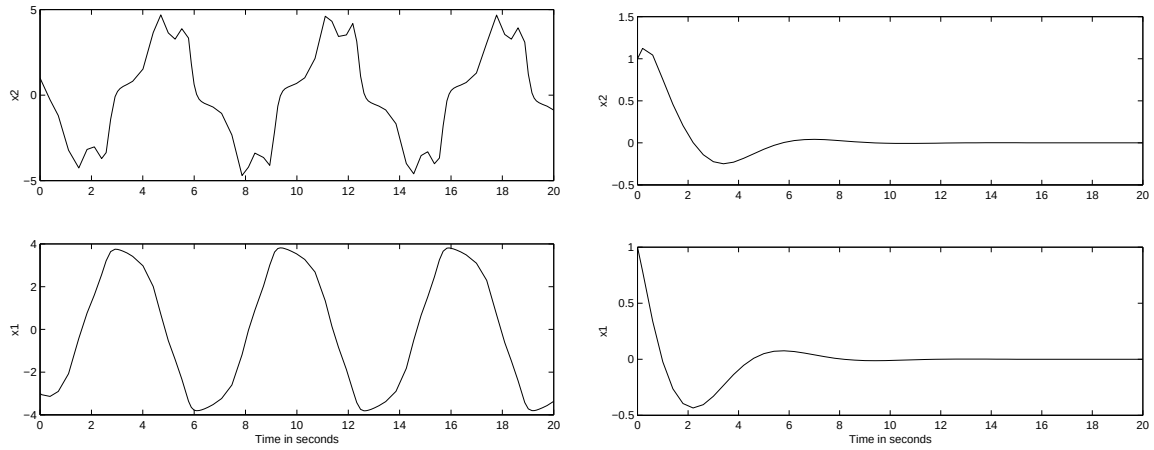
$23^{rd}$ **May 1998**
Consider the system

$$\dot{x_1} = -x_2 \tag{23}$$
$$\dot{x_2} = x_1 - x_2\left(1 - x_1^2 + 0.1\,x_1^4\right) + u, \tag{24}$$

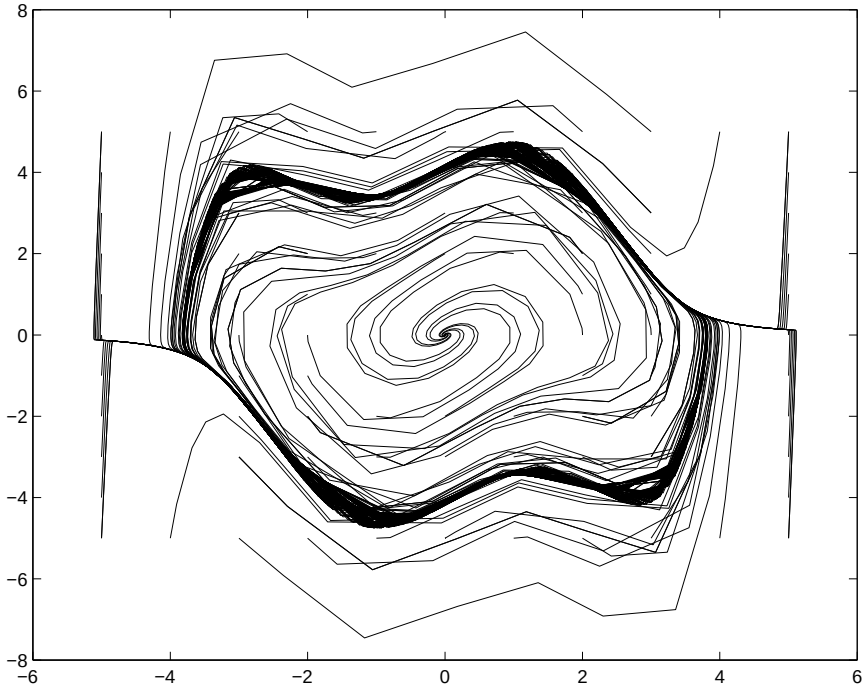
let $u = 0$ first and plot the results in Figure 15. In Figure 15 (c) the initial conditions used were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\} \bigcup \{-3 \leq \mathcal{I} \leq 3\} \times \{-3 \leq \mathcal{I} \leq 3\} \bigcup \{-2 \leq \mathcal{I} \leq 2\} \times \{-2 \leq \mathcal{I} \leq 2\}$. With the subset of the initial conditions used $(x_1, x_2) \in \{-2 \leq \mathcal{I} \leq 2\} \times \{-2 \leq \mathcal{I} \leq 2\}$ the results converged to the origin, the rest of the set converge the results onto limit cycles. Also notice that if the system equation is slightly changed into

$$\dot{x_1} = -x_2 \tag{25}$$
$$\dot{x_2} = x_1 - x_2\left(1 - x_1^2 - 0.1\,x_1^4\right) \tag{26}$$

the system will have a singularity and will no longer be a simulateable system.



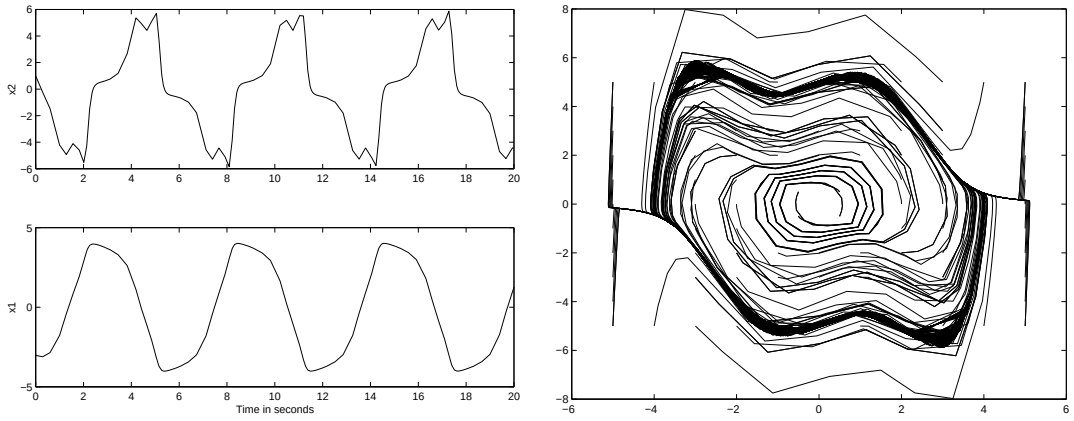

(a) (b)



(c)

Figure 15: *(a) The state variables ($t_{simulation} = 20$ sec, $t_{CPU} = 0.0300$ sec, Initial conditions are $(x_1, x_2) = (-3, 1)$) (b) The state variables ($t_{simulation} = 20$ sec, $t_{CPU} = 0.0200$ sec, Initial conditions are $(x_1, x_2) = (1, 1)$), and (c) the state plane ($t_{simulation} = 20$ sec, $t_{CPU,average} = 0.0222$ sec, Initial conditions are $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\} \bigcup \{-3 \leq \mathcal{I} \leq 3\} \times \{-3 \leq \mathcal{I} \leq 3\} \bigcup \{-2 \leq \mathcal{I} \leq 2\} \times \{-2 \leq \mathcal{I} \leq 2\}$), simulated on Hayate*

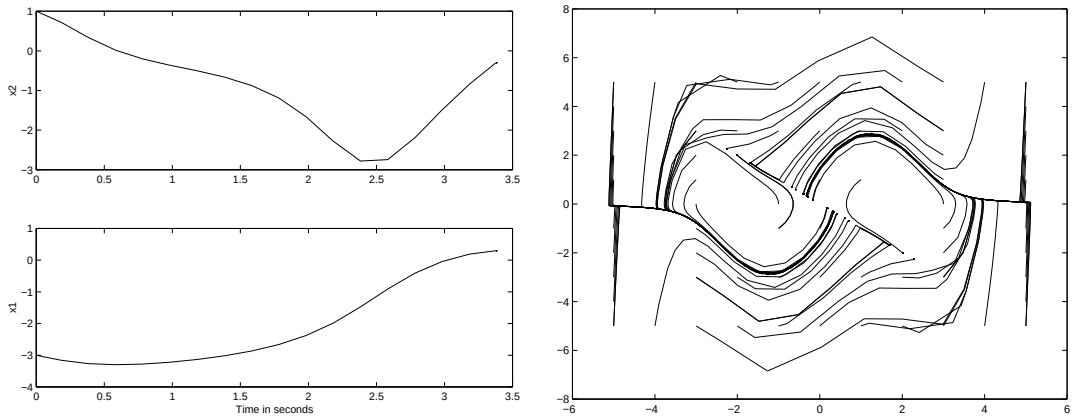
Then introduce an input $u = \mathrm{sgn}(x_1 + x_2)$ and plot the results shown in Figure 16. In Figure 16 (b), as contrasted to Figure 15 (c), all the set of the initial conditions tried converged the trajectories onto limit cycles.




(a) (b)

Figure 16: *(a) The state variables ($t_{simulation}$ = 20 sec, $t_{CPU}$ = 0.0400 sec, Initial conditions are $(x_1, x_2) = (-3, 1)$) (b) the state plane ($t_{simulation}$ = 20 sec, $t_{CPU,average}$ = 0.0335 sec, Initial conditions are $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\} \bigcup \{-3 \leq \mathcal{I} \leq 3\} \times \{-3 \leq \mathcal{I} \leq 3\} \bigcup \{-2 \leq \mathcal{I} \leq 2\} \times \{-2 \leq \mathcal{I} \leq 2\} \bigcup \{-0.5 \leq \mathcal{I} \leq 0.5\} \times \{-0.5 \leq \mathcal{I} \leq 0.5\}$), simulated on Hayate*

Then introduce an input $u = -2\ \mathrm{sgn}(x_1 + x_2)$ and plot the results shown in Figure 17. In Figure 17 (b), as contrasted to Figure 15 (c) and Figure 16 (b), all the set of the initial conditions tried converged the trajectories onto the $x_1 + x_2 = 0$ hyperplane where sliding occurs. An actual simulation would have taken too much computer resources, namely computer time and memory, to be practical therefore manual interruptions were used in order to try to reduce these overheads. Therefore neither the $t_{simulation} = 10$ sec nor the set of initial conditions planned has been carried out.




(a) (b)

Figure 17: *(a) The state variables ($t_{CPU} = 1.800$ sec, Initial conditions are $(x_1, x_2) = (-3, 1)$) (b) the state plane ($t_{CPU} = 76.7100$ sec, Initial conditions as planned were $(x_1, x_2) \in \{-5 \leq \mathcal{I} \leq 5\} \times \{-5 \leq \mathcal{I} \leq 5\} \bigcup \{-3 \leq \mathcal{I} \leq 3\} \times \{-3 \leq \mathcal{I} \leq 3\} \bigcup \{-2 \leq \mathcal{I} \leq 2\} \times \{-2 \leq \mathcal{I} \leq 2\}$), simulated on Hayate*

# Chaotic model

$20^{th}$ **May 1998**
Consider the chaotic model

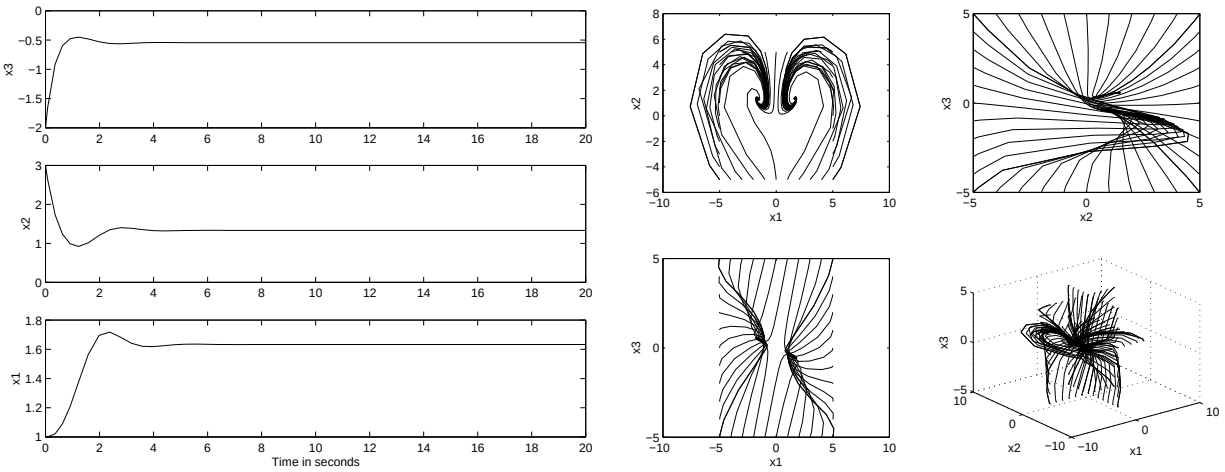
$$\dot{x_1} = x_1 - x_1 x_2 - x_3 \tag{27}$$
$$\dot{x_2} = x_1^2 - a\, x_2 \tag{28}$$
$$\dot{x_3} = b\, x_1 - c\, x_3 + u \tag{29}$$

of which the results when $a = 2, b = 1, c = 3$, and $u = 0$ is shown in Figure 18.

The initial conditions used from Figure 18 to 20 is $(x_1, x_2, x_3) = (1, 3, -2)$ for all of the three figures (a)'s, and

$$\begin{aligned}(x_1, x_2, x_3) \in\ & \{-5 \le \mathcal{I} \le 5\} \times \{-5 \le \mathcal{I} \le 5\} \times \{1\} \bigcup \\ & \{1\} \times \{-5 \le \mathcal{I} \le 5\} \times \{-5 \le \mathcal{I} \le 5\} \bigcup \\ & \{-5 \le \mathcal{I} \le 5\} \times \{1\} \times \{-5 \le \mathcal{I} \le 5\}\end{aligned}$$

for all of the three figures (b)'s. The CPU time is 0.01 seconds for all the three (a) figures, and the averaged CPU time for each initial points in the three (b) figures are 0.0146, 0.0183, and 95.01 seconds for Figure 18 (b), 19 (b), and 20 (b) respectively, all simulation being done on Hayate with simulation time 20 seconds.



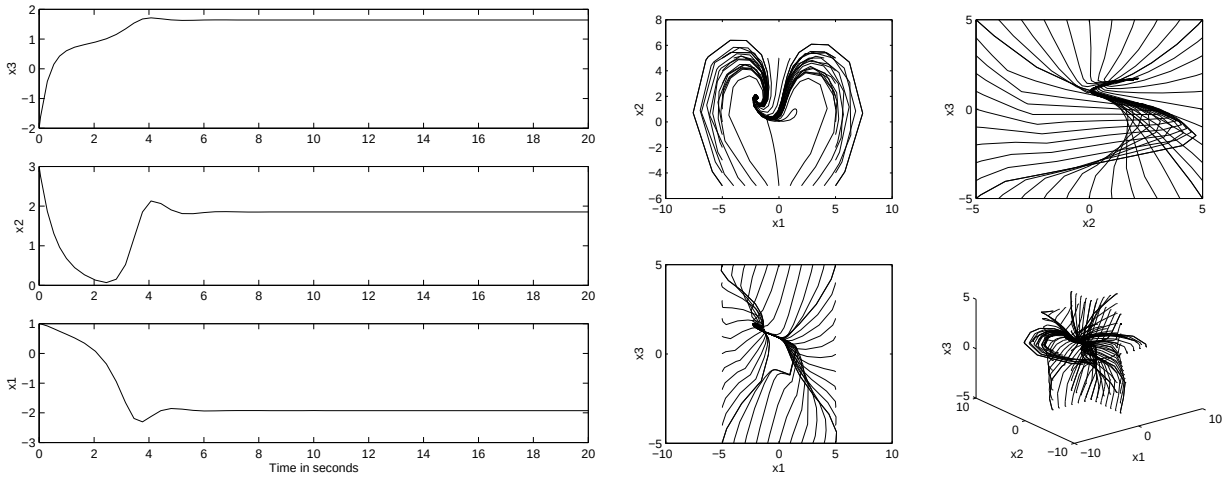

(a) (b)

Figure 18: *(a) State variables, (b) state plane for the chaotic model*

Let the input be

$$u = 3\,\mathrm{sgn}(x_1 + x_2 + x_3)$$

be an input to the system and plot Figure 19.



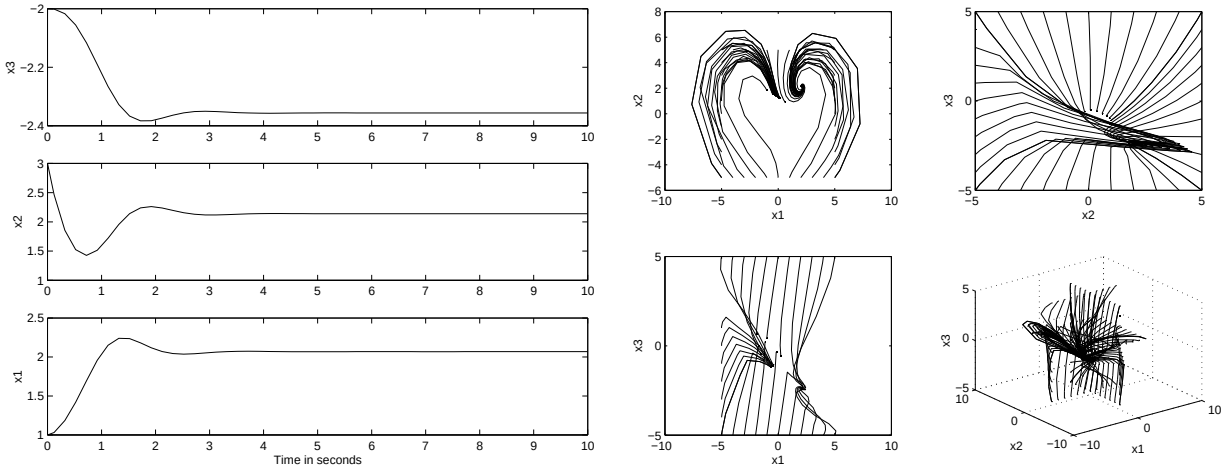

(a) (b)

Figure 19: *(a) State variables, (b) state plane for the chaotic model*

Let the input be

$$u = -5\ \mathrm{sgn}(x_1 + x_2 + x_3)$$

be an input to the system and plot Figure 20. Here $k = -5$ as contrasted to Figure 18 where $k = 0$ and Figure 19 where $k = 3$. In order to reduce the file size manual interruptions were applied where necessary when simulated using Matlab. Therefore both the $t_{simulation} = 10$ sec and the set of all initial conditions planned for the simulation has not been completed.




(a) (b)

Figure 20: *(a) State variables, (b) state plane for the chaotic model*

# Third order system with a toroidal trajectories

$17^{th}$ **May 1998**
Consider the system

$$\dot{x_1} = \upsilon x_2 - x_1 x_3 \tag{30}$$
$$\dot{x_2} = -\upsilon x_1 - x_2 x_3 \tag{31}$$
$$\dot{x_3} = \ln \sqrt{(x_1^2 + x_2^2)} + u \tag{32}$$

which when $\upsilon = 14$ has trajectories on a toroidal surfaces. Figure 21 and 22 shows the result with $u = 0$, $\upsilon = 14$, the initial condition at

$$(x_1, x_2, x_3) = (1, 1, 1)\,,$$

$t_{simulation} = 20$ sec, used $t_{cpu} = 0.0800$ sec in the case of Figure 21 $t_{cpu,average} = 0.1400$ sec in the case of Figure 22 simulated on Fubuki with no input for Figure 21 while

$$u = -3\,\mathrm{sgn}(x_1 + x_2 + x_3)$$

for Figure 22 In both figures the top left is the top view, top right side view, bottom left front view and bottom right an isometric.

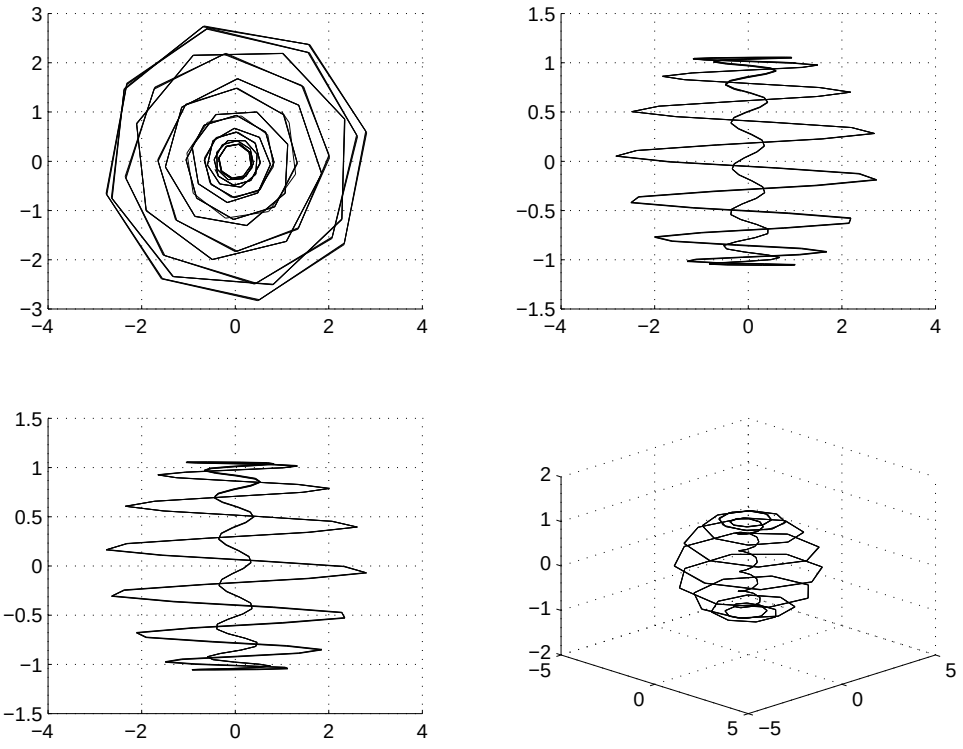



Figure 21: *State plane of the system with a toroidal trajectories*

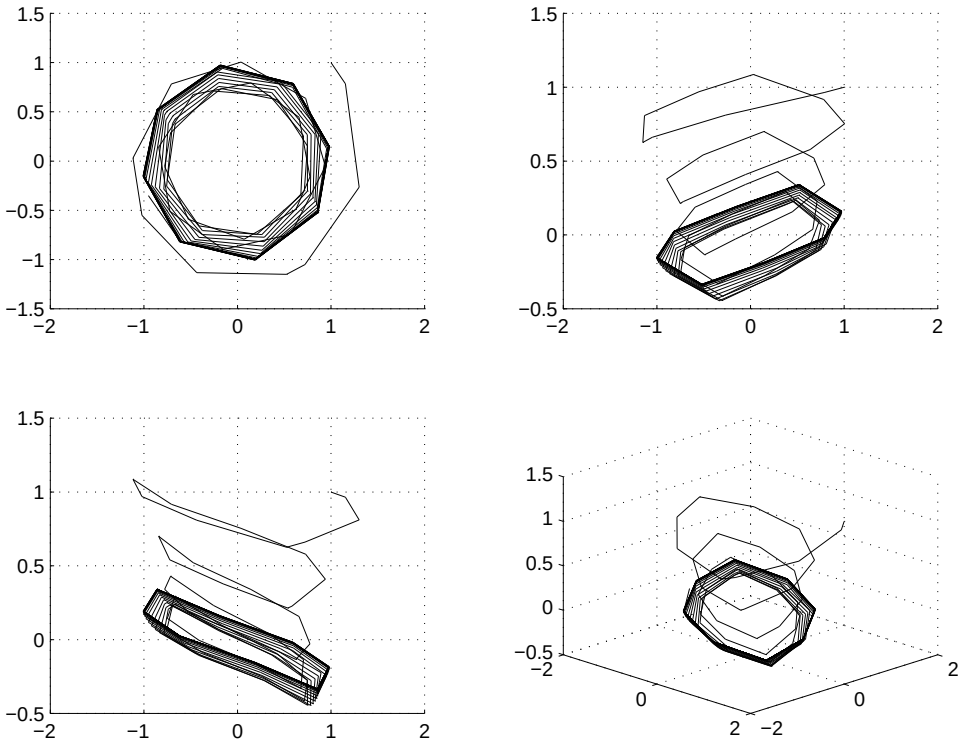



Figure 22: *State plane of the system with a toroidal trajectories*

Both Figure 23 and 24 shows the result when

$$u = -9\,\mathrm{sgn}(x_1 + x_2 + x_3),\ \text{and}\ \upsilon = 14$$

simulated on Hayate using $t_{cpu} = 0.1400$ sec. The initial conditions are

$$(x_1, x_2, x_3) = (1, 1, 1)$$

for Figure 23 and

$$(x_1, x_2, x_3) = (-5, -4, 5)\,, (2, 2, -3)\,, (-4, 1, -3)$$

for Figure 24.

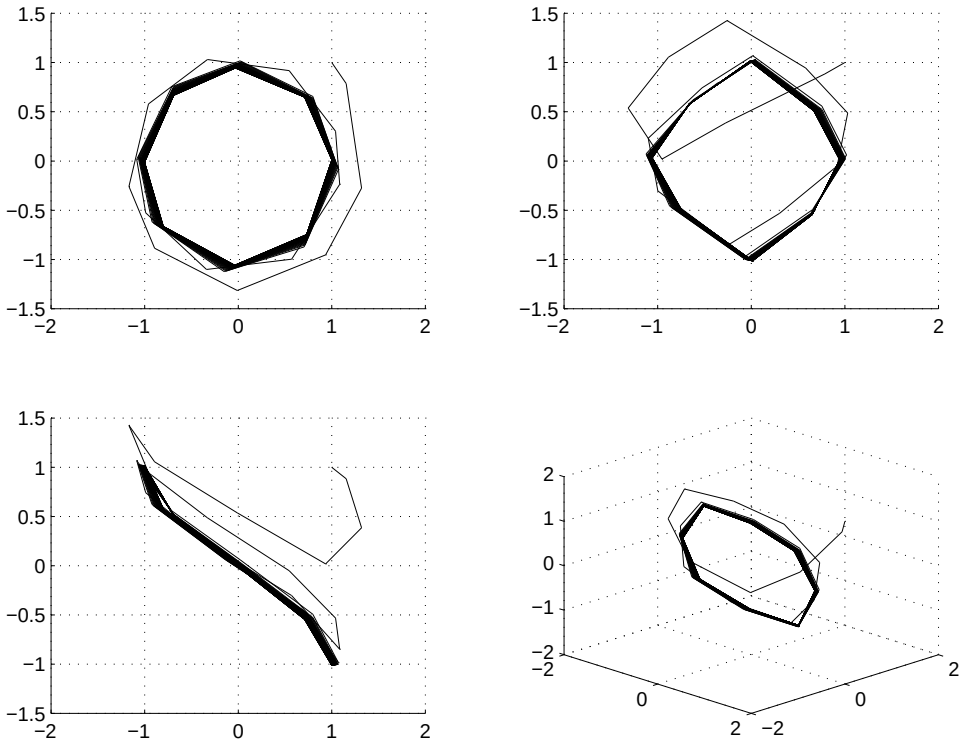


Figure 23: *State plane of the system with a toroidal trajectories*

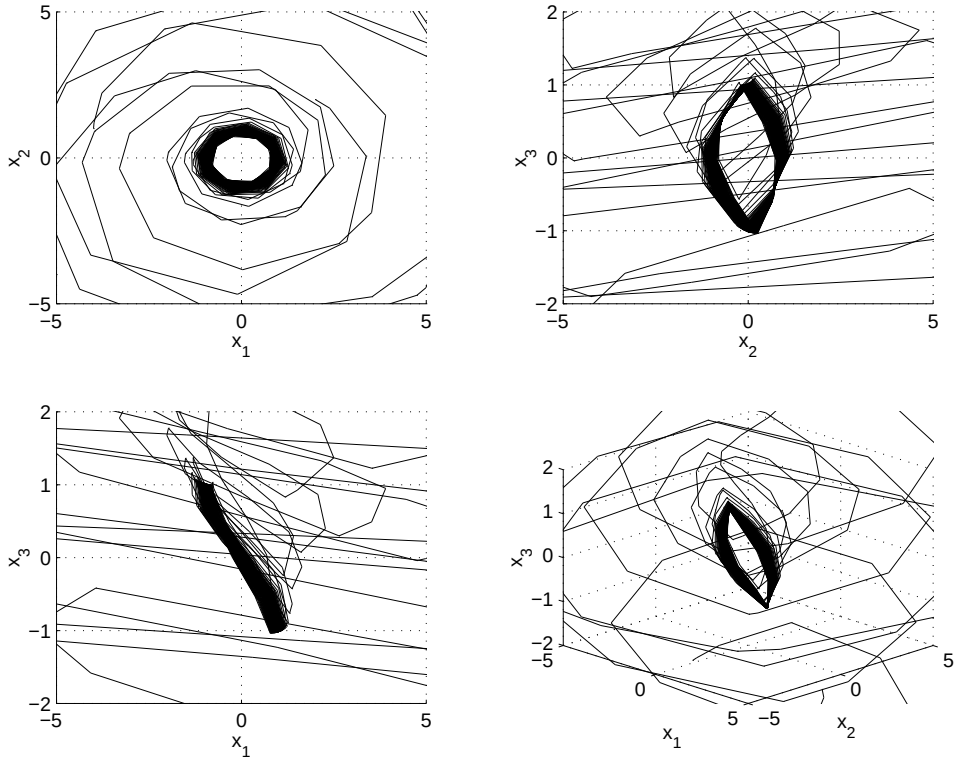



Figure 24: *State plane of the system with a toroidal trajectories*

When

$$u = 3\,\mathrm{sgn}(x_1 + x_2 + x_3)$$

the trajectory ran away and expanded to great values of $x_1$ and $x_2$ as shown in Figure 25 ($\upsilon = 14$, $t_{cpu} = 0.1500$ sec simulated on Hayate.)

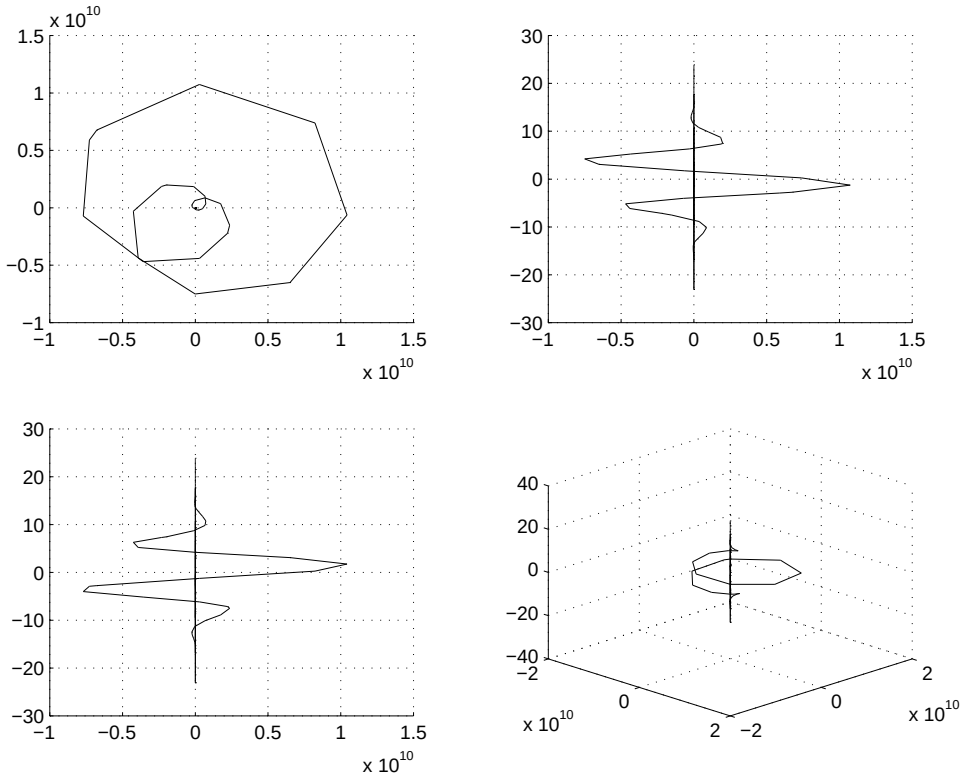



Figure 25: *State plane of the system with a toroidal trajectories*

# Coupled dynamos

$16^{th}$ **May 1998**
Consider a system of coupled dynamos system which posseses chaotic behaviour.

$$\begin{aligned}
\dot{x_1} &= -\mu_1 x_1 + \omega_1 x_2 && (33)\\
\dot{x_2} &= -\mu_2 x_2 + \omega_2 x_1 && (34)\\
\dot{\omega_1} &= q_1 - \eta_1 \omega_1 - x_1 x_2 && (35)\\
\dot{\omega_2} &= q_2 - \eta_2 \omega_2 - x_1 x_2 + u && (36)
\end{aligned}$$

where $q_1$, $q_2$ are torques applied to the rotor and $\mu_1$, $\mu_2$, $\eta_1$, $\eta_2$ are dissipative effects $\mu_1, \mu_2, \eta_1, \eta_2 \in \mathcal{I}^+$. Let $u = 0$. Simulated with simulation time $t_{simulation} = 10$ sec, Figure 26 and Figure 27 are the result from simulation of this system when

$$\mu_1 = 0.2,\ \mu_2 = 0.3,\ \eta_1 = 0.15,\ \eta_2 = 0.4,\ q_1 = 1,\ q_2 = 1.5,$$

while Figure 28 and Figure 29 are the result when

$$\mu_1 = 1,\ \mu_2 = 2,\ \eta_1 = 0.1,\ \eta_2 = 0.5,\ q_1 = 2,\ q_2 = -1.$$

The average CPU time $t_{CPU,average}$ are 0.0200, 0.0214, 0.0200, and 0.0203 sec for Figure 26, 27, 28, and 29 respectively.

The initial condition for Figure 26 and 28 is at

$$(x_1, x_2, \omega_1, \omega_2) = (1, 1, 1, 1)\,,$$

while for Figure 27 and 29 it is

**top left corner**

$$\begin{aligned}
(x_1, x_2, \omega_1, \omega_2) &= \{-5 \le \mathcal{I} \le 5\} \times \{-5 \le \mathcal{I} \le 5\} \times \{1\} \times \{1\} \bigcup \\
&\quad \{-5 \le \mathcal{I} \le 5\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\} \times \{1\} \times \{1\}
\end{aligned}$$

**top right corner**

$$\begin{aligned}
(x_1, x_2, \omega_1, \omega_2) &= \{1\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\} \times \{-5 \le \mathcal{I} \le 5\} \times \{1\} \bigcup \\
&\quad \{1\} \times \{-5 \le \mathcal{I} \le 5\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\} \times \{1\}
\end{aligned}$$

**bottom left corner**

$$\begin{aligned}
(x_1, x_2, \omega_1, \omega_2) &= \{1\} \times \{1\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\} \times \{-5 \le \mathcal{I} \le 5\} \bigcup \\
&\quad \{1\} \times \{1\} \times \{-5 \le \mathcal{I} \le 5\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\}
\end{aligned}$$

**bottom right corner**

$$\begin{aligned}
(x_1, x_2, \omega_1, \omega_2) &= \left\{-5 \le \mathcal{I}^+ \le 5\right\} \times \{1\} \times \{1\} \times \{-5 \le \mathcal{I} \le 5\} \bigcup \\
&\quad \{-5 \le \mathcal{I} \le 5\} \times \{1\} \times \{1\} \times \left\{-5 \le \mathcal{I}^+ \le 5\right\}
\end{aligned}$$

for Figure 27, Figure 29 and Figure 31.

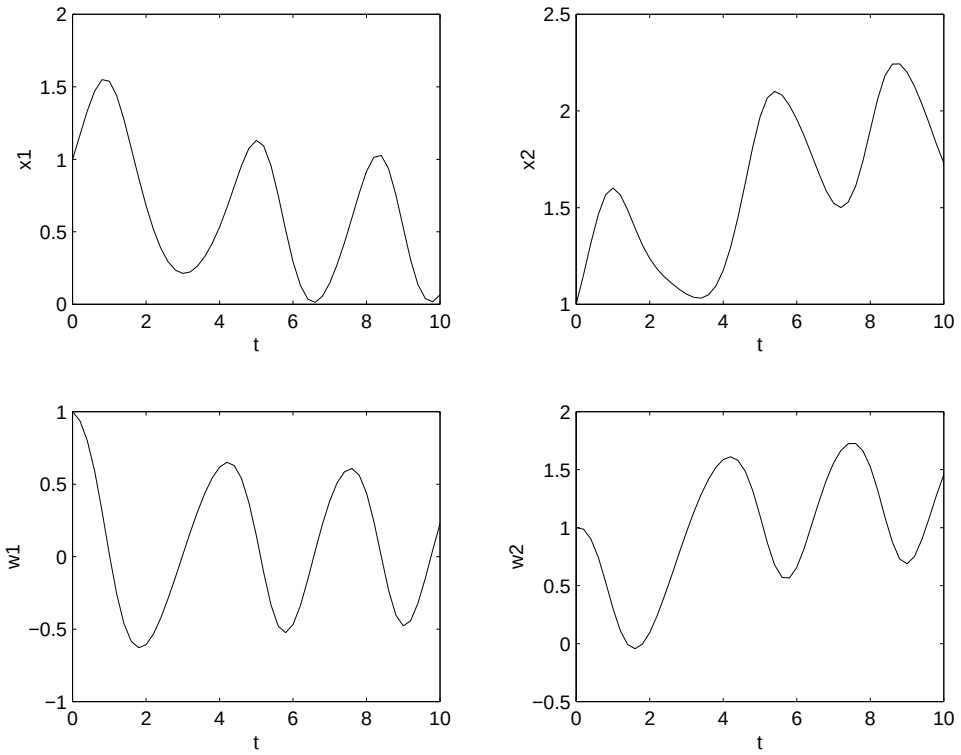



Figure 26: *State variables of a coupled dynamos system.*

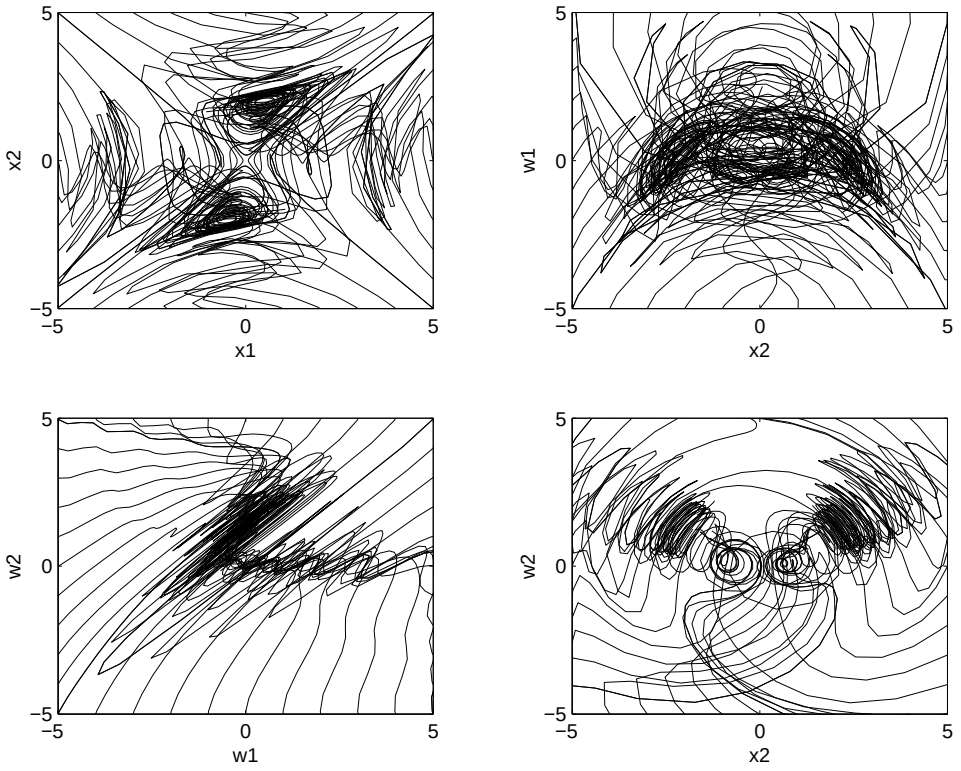



Figure 27: *State planes of a coupled dynamos system.*

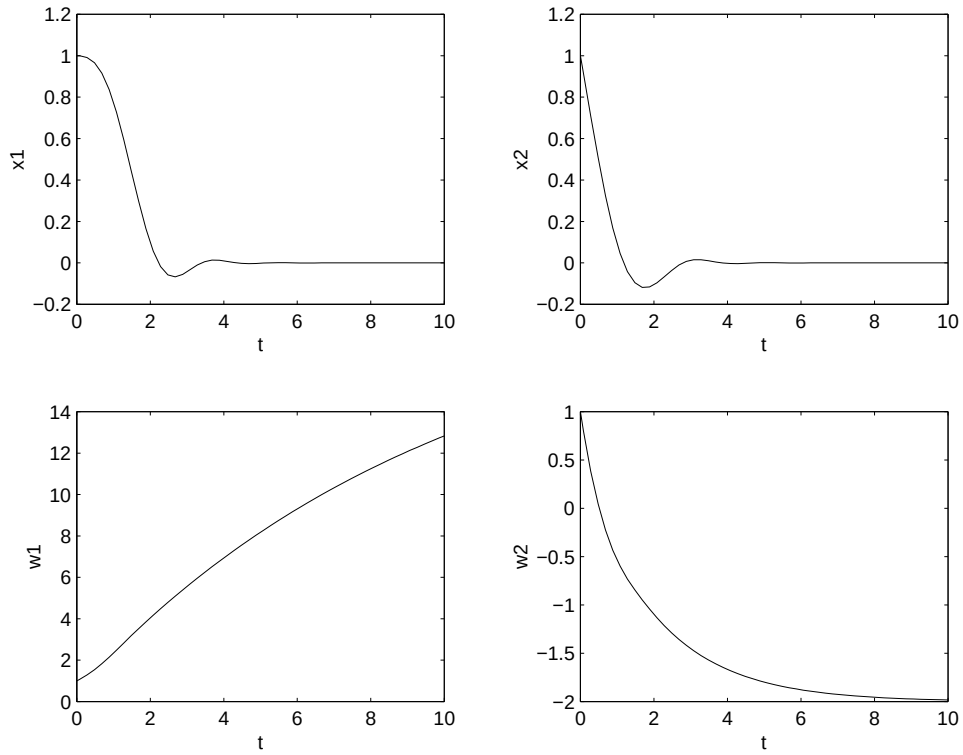



Figure 28: *State variables of a coupled dynamos system.*

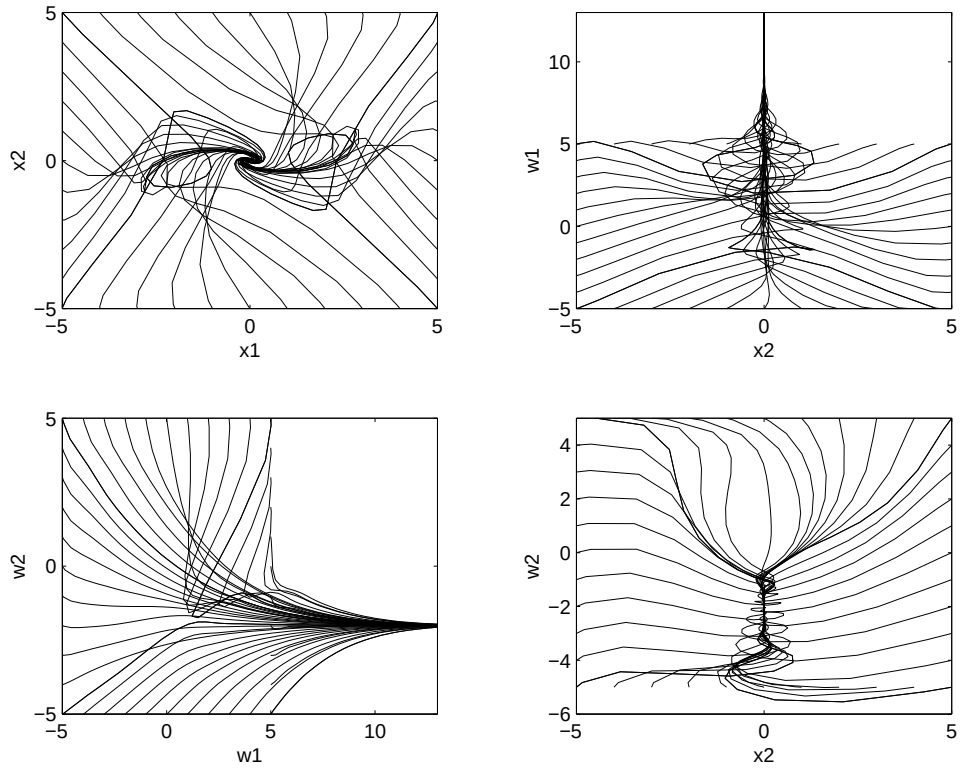



Figure 29: *State planes of a coupled dynamos system.*

Then Figure 28 and Figure 29 was obtained with an input

$$u = -5\,\mathrm{sgn}\left(x_1 + x_2 + w_1 + w_2\right),$$

the parameters

$$\mu_1 = 1,\ \mu_2 = 2,\ \eta_1 = 0.1,\ \eta_2 = 0.5,\ q_1 = 2,\ q_2 = -1,$$

$t_{CPU} = 1.8700$ sec for Figure 28, $t_{CPU} = 335.71$ sec for Figure 29, the initial condition is

$$(x_1, x_2, \omega_1, \omega_2) = (1, 1, 1, 1)$$

simulated on Hayate.

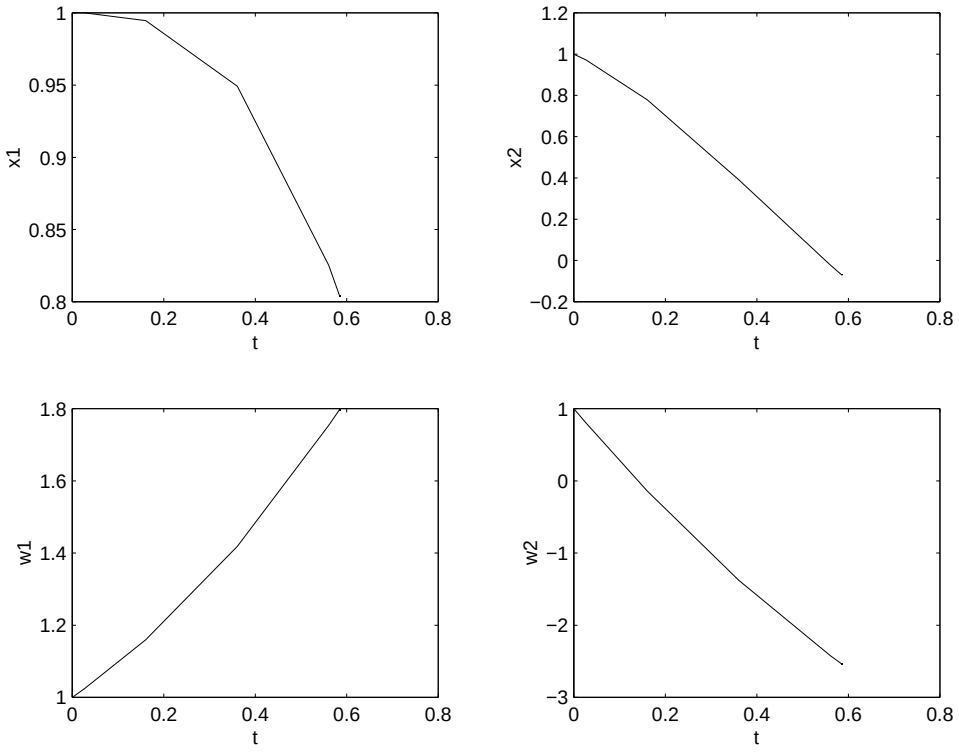



Figure 30: *State variables of a coupled dynamos system.*

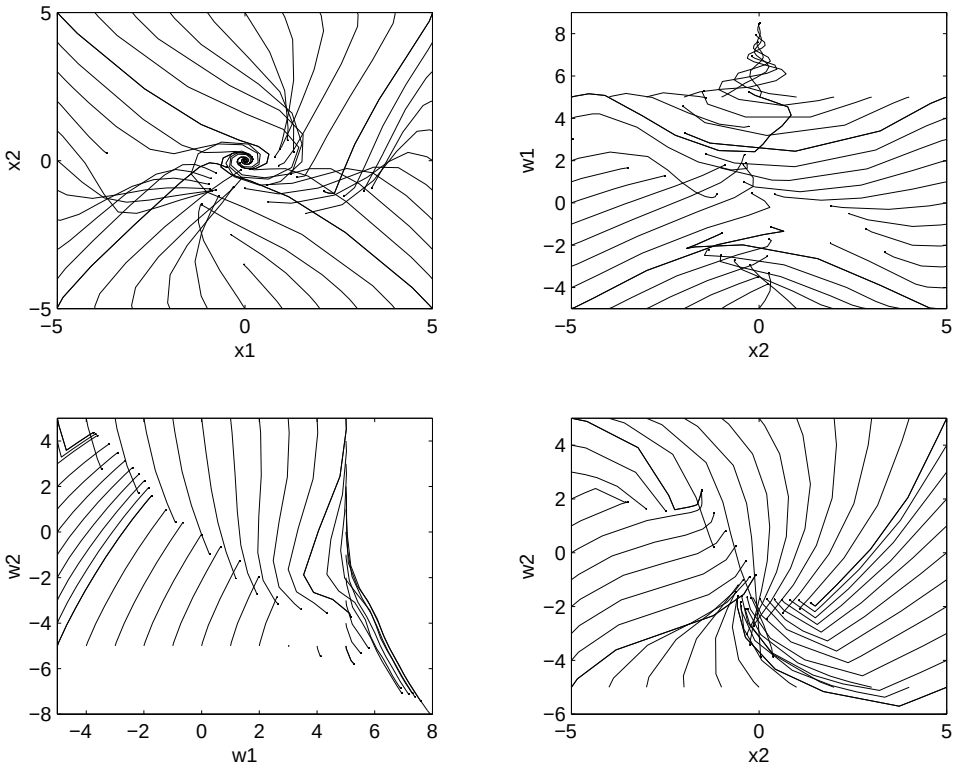



Figure 31: *State planes of a coupled dynamos system.*

# Net National Product

The NNP optimization problem studied by Dasgupta [Das93] based on the following model.

$$\frac{dK_1}{dt} = F(K_1, L_1) - C - Q - R \tag{37}$$
$$\frac{dK_2}{dt} = G(K_3) - A[K_4, F(K_1, L_1)] \tag{38}$$
$$\frac{dK_3}{dt} = H(L_2) - X \tag{39}$$
$$\frac{dK_4}{dt} = Q \tag{40}$$
$$X = N(K_3, L_3) \tag{41}$$
$$Z = J(R, A(K_4, F(K_1, L_1))) \tag{42}$$

where

$K_1$ = stock of a multi-purposed, man-made, perfectly durable capital goods
$L_1$ = the labour effort combined with this goods
$Y$ = $F(K_1, L_1)$ is the flow of output and assumed to be a concave function
$K_2$ = clean air (a environmental resource stock)
$K_3$ = forests (a environmental resource stock)
$K_4$ = the stock of the defensive capital against pollution
$C$ = rate of consumption of the output $Y$
$Q$ = the expenditure on the accumulation of $K_4$
$R$ = rate of expenditure of the output $Y$ in order to counter the damages to the flow of environmental amenities
$G(\cdot)$ = the rate of regeneration of the environmental stock
$P$ = $A(K_4, Y)$ is the emission of pollutants, $A$ is a convex function
$H(\cdot)$ = the rate of regeneration of forests by labour effort
$L_2$ = the labour effort used in regenerating forests
$X$ = $N(K_3, L_3)$ is the rate of consumption of fuelwood, $N$ is a concave function increasing with $K_3$ and $L_3$
$L_3$ = the labour effort to collect fuelwood for consumption
$Z$ = $J(R, P)$ is the flow of amenities, $J$ is increasing in $R$ and decreasing in $P$.

# Bibliography


[Das93] Partha Dasgupta. *An inquiry into well-being and destitution.* Oxford University Press, 1993.

[Kha96] Hassan K. Khalil. *Nonlinear systems.* Prentice Hall, 2 edition, 1996.